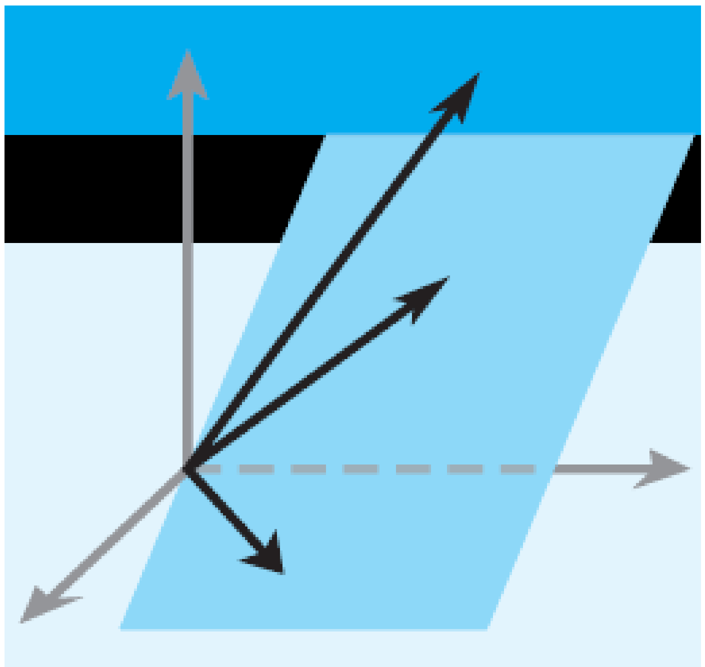
Elementary Linear Algebra



Chapter 4

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Chapter 4
General Vector Spaces

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- 4.2 Subspaces
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Section 4.1 Vector Space Axioms

DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

To Show that a Set with Two Operations is a Vector Space

1. Identify the set V of objects that will become vectors.
2. Identify the addition and scalar multiplication operations on V.
3. Verify Axioms 1(***closure under addition***) and 6 (***closure under scalar multiplication***) ; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V.
4. Confirm that Axioms 2,3,4,5,7,8,9 and 10 hold.

Section 4.2 Subspaces

DEFINITION 1 A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions hold.*

(a) *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.*

(b) *If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.*

The 'smallest' subspace of a vector space V

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

THEOREM 4.2.3 *If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:*

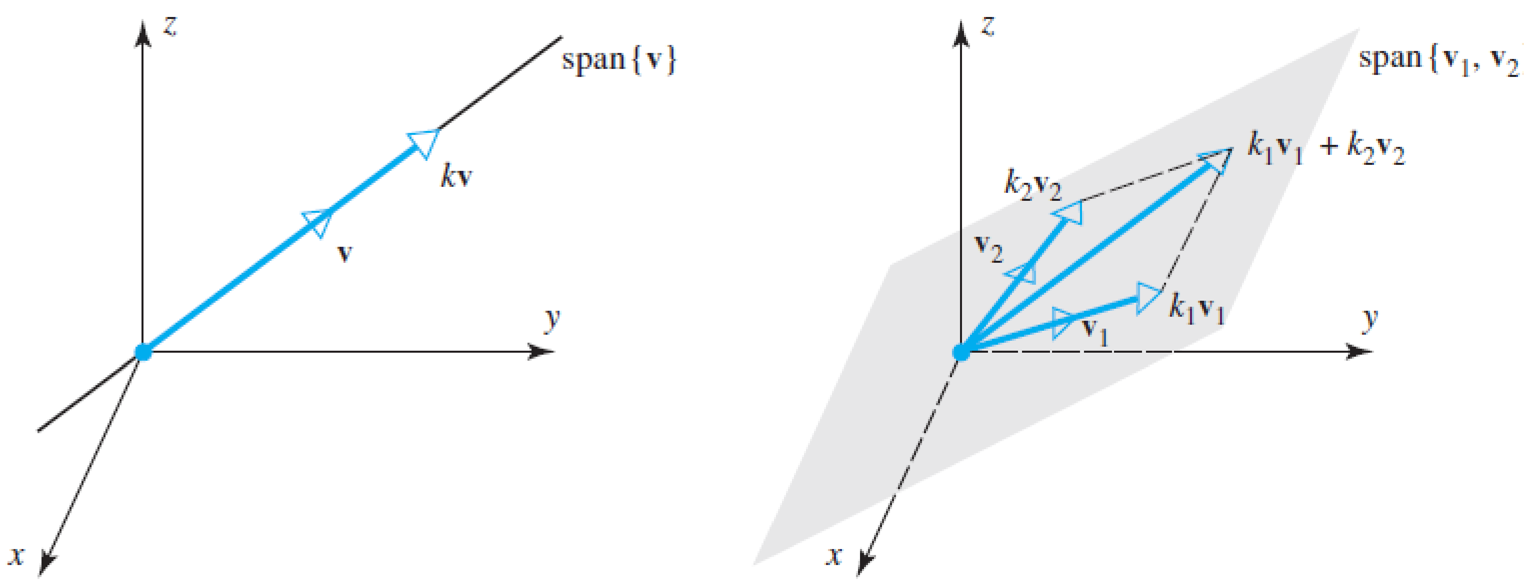
(a) *The set W of all possible linear combinations of the vectors in S is a subspace of V.*

(b) *The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.*

The span of S

DEFINITION 3 The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the ***span of S***, and we say that the vectors in S ***span*** that subspace. If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$, then we denote the span of S by

$$\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad \text{span}(S)$$



(*a*) Span$\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

(*b*) Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Section 4.3 Linear Independence

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely,

$$k_1 = 0, \quad k_2 = 0, \ldots, \quad k_r = 0$$

We call this the ***trivial solution***. If this is the only solution, then S is said to be a ***linearly independent set***. If there are solutions in addition to the trivial solution, then S is said to be a ***linearly dependent set***.

Linearly independence

THEOREM 4.3.1 *A set S with two or more vectors is*

(*a*) *Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

(*b*) *Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.*

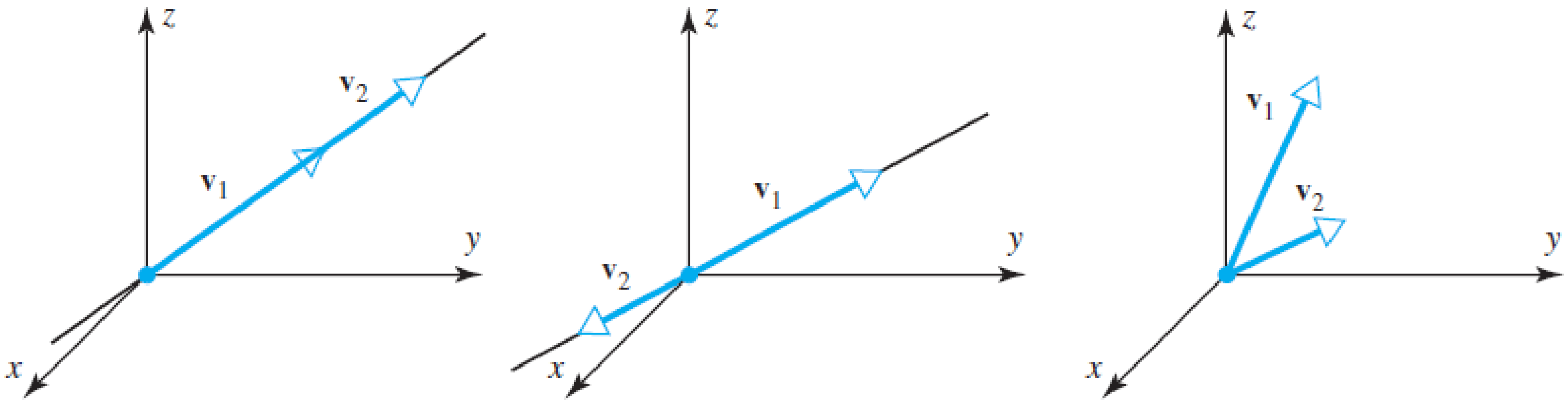
THEOREM 4.3.2

(*a*) *A finite set that contains $\mathbf{0}$ is linearly dependent.*

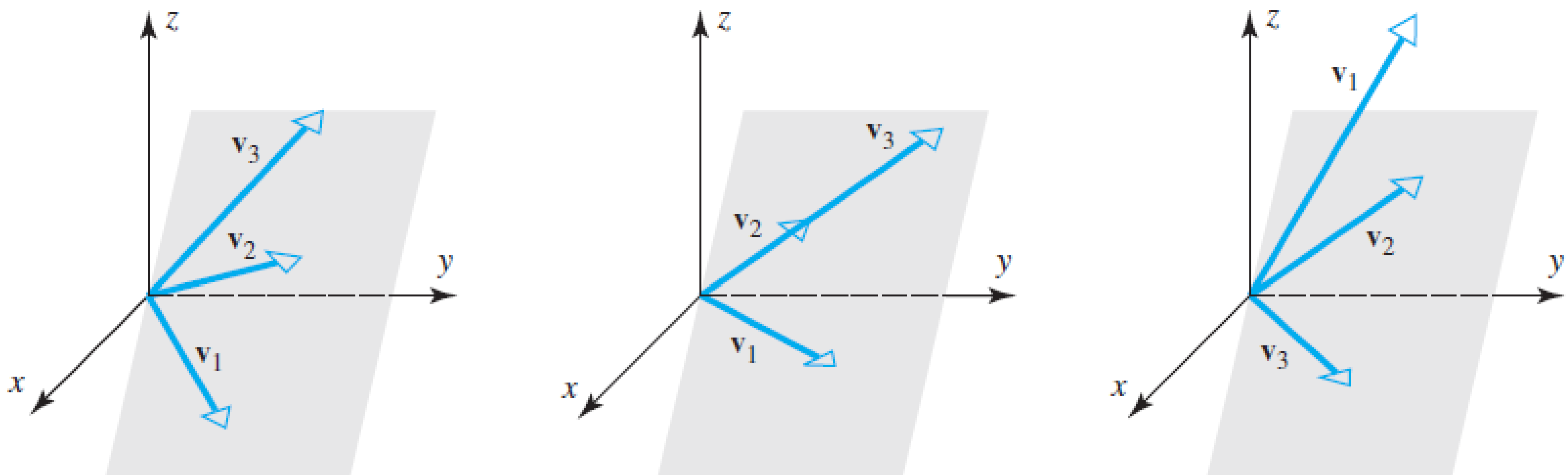
(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.*

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Linear Independence in R^2 and R^3



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

The Wronskian

DEFINITION 2 If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$.

THEOREM 4.3.4 *If the functions* $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ *have* $n-1$ *continuous derivatives on the interval* $(-\infty, \infty)$, *and if the Wronskian of these functions is not identically zero on* $(-\infty, \infty)$, *then these functions form a linearly independent set of vectors in* $C^{(n-1)}(-\infty, \infty)$.

Section 4.4 Coordinates and Basis

DEFINITION 1 If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a finite set of vectors in V, then S is called a ***basis*** for V if the following two conditions hold:

(a) S is linearly independent.

(b) S spans V.

THEOREM 4.4.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

The coordinate vector

DEFINITION 2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the ***coordinates*** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the ***coordinate vector of* $\mathbf{v}$ *relative to* S**; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n) \tag{6}$$

Section 4.5 Dimension

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

$\dim(R^n) = n$	**The standard basis has n vectors.**
$\dim(P_n) = n + 1$	**The standard basis has $n + 1$ vectors.**
$\dim(M_{mn}) = mn$	**The standard basis has mn vectors.**

Plus / Minus Theorem

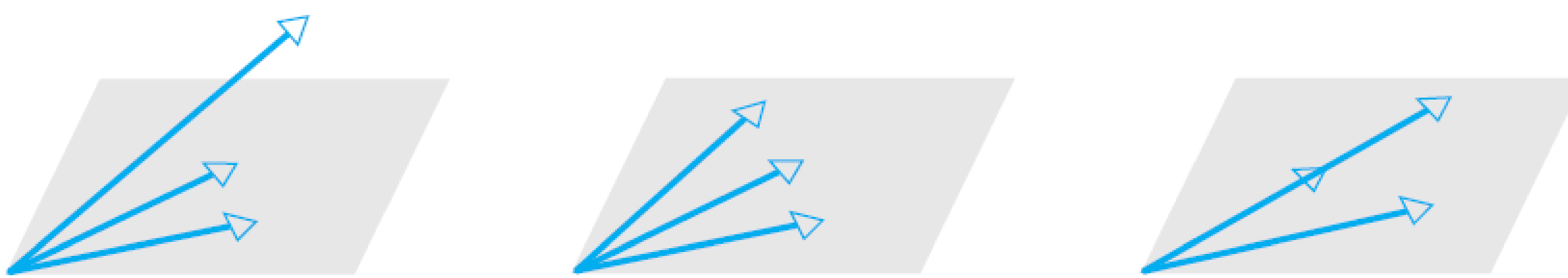
THEOREM 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.*

(b) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,*

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$



The vector outside the plane can be adjoined to the other two without affecting their linear independence.

Any of the vectors can be removed, and the remaining two will still span the plane.

Either of the collinear vectors can be removed, and the remaining two will still span the plane.

THEOREM 4.5.4 *Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.*

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(*a*) *W is finite-dimensional.*

(*b*) $\dim(W) \leq \dim(V)$.

(*c*) *$W = V$ if and only if* $\dim(W) = \dim(V)$.

Section 4.6 Change of Basis

The Change-of-Basis Problem If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Solution of the Change-of-Basis Problem If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} \tag{7}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}'_1]_B, \quad [\mathbf{u}'_2]_B, \ldots, \quad [\mathbf{u}'_n]_B \tag{8}$$

Transition Matrices

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

THEOREM 4.6.1 *If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.*

Computing the transition matrix

A Procedure for Computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

This procedure is captured in the following diagram.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{14}$$

Section 4.7 Row Space, Column Space, and Null Space

DEFINITION 1 For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n that are formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the ***column vectors*** of A.

Row, column and null spaces

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the ***null space*** of A.

Systems of linear equations

Question 1. What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A?

Question 2. What relationships exist among the row space, column space, and null space of a matrix?

THEOREM 4.7.1 *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

THEOREM 4.7.2 *If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{3}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

A basis for span (S)

Problem Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in R^n, find a subset of these vectors that forms a basis for span(S), and express those vectors that are not in that basis as a linear combination of the basis vectors.

Basis for Span(S)

Step 1. Form the matrix A having vectors in $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ as column vectors.

Step 2. Reduce the matrix A to reduced row echelon form R.

Step 3. Denote the column vectors of R by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

Step 4. Identify the columns of R that contain the leading 1's. The corresponding column vectors of A form a basis for span(S).

Step 5. Obtain a set of dependency equations by expressing each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that do contain leading 1's.

Step 6. Replace the column vectors of R that appear in the dependency equations by the corresponding column vectors of A.

This completes the second part of the problem.

Section 4.8 Rank, Nullity, and the Fundamental Matrix Spaces

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\operatorname{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\operatorname{nullity}(A)$.

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$rank(A) + nullity(A) = n$$

THEOREM 4.8.3 *If A is an $m \times n$ matrix, then*

(*a*) $rank(A) =$ *the number of leading variables in the general solution of* $A\mathbf{x} = \mathbf{0}$.

(*b*) $nullity(A) =$ *the number of parameters in the general solution of* $A\mathbf{x} = \mathbf{0}$.

THEOREM 4.8.4 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The column vectors of A are linearly independent.*

(i) *The row vectors of A are linearly independent.*

(j) *The column vectors of A span R^n.*

(k) *The row vectors of A span R^n.*

(l) *The column vectors of A form a basis for R^n.*

(m) *The row vectors of A form a basis for R^n.*

(n) *A has rank n.*

(o) *A has nullity 0.*

Fundamental Spaces of Matrix A

- Row space of A
- Null space of A
- Column space of A
- Null space of A^T

THEOREM 4.8.10 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of A is I_n.*

(d) A *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The column vectors of A are linearly independent.*

(i) *The row vectors of A are linearly independent.*

(j) *The column vectors of A span R^n.*

(k) *The row vectors of A span R^n.*

(l) *The column vectors of A form a basis for R^n.*

(m) *The row vectors of A form a basis for R^n.*

(n) A *has rank* n.

(o) A *has nullity* 0.

(p) *The orthogonal complement of the null space of A is R^n.*

(q) *The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.*

Section 4.9 Matrix Transformations from R^n to R^m

DEFINITION 1 If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a ***transformation*** from V to W or that f ***maps*** V to W, which we denote by writing

$$f: V \to W$$

In the special case where $V = W$, the transformation is also called an ***operator*** on V.

THEOREM 4.9.1 *For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k:*

(*a*) $T_A(\mathbf{0}) = \mathbf{0}$

(*b*) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(*c*) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(*d*) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Finding the standard matrix for a matrix transformation

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n in column form.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

Table 1

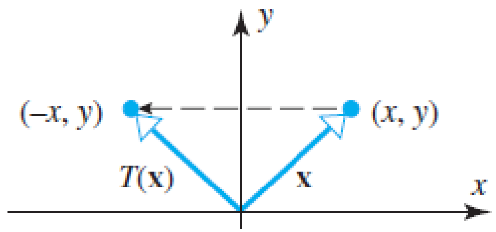
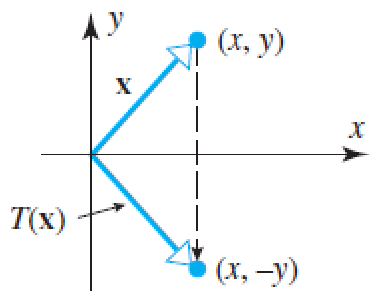
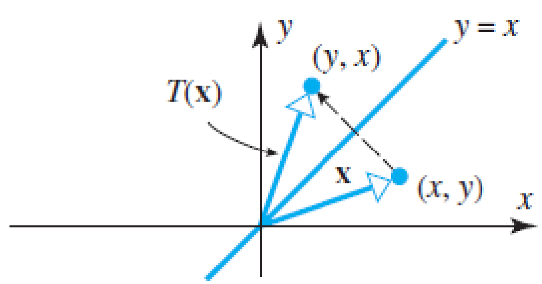
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 2

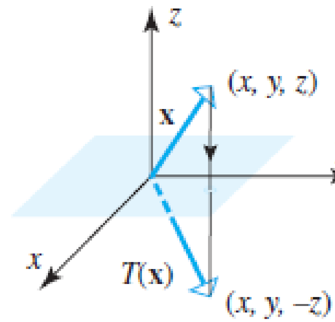
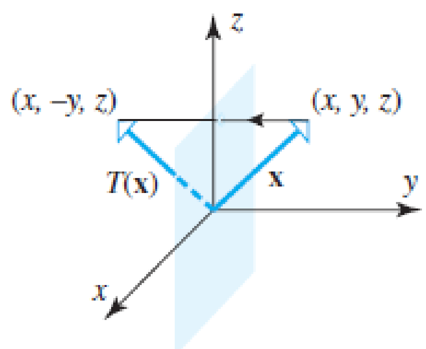
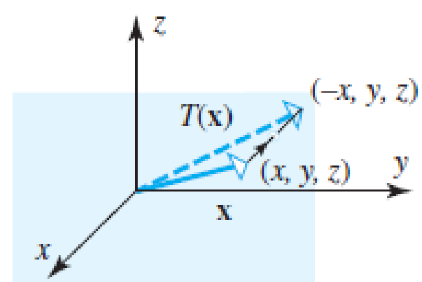
<table>
<tr><th>Operator</th><th>Illustration</th><th>$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$</th><th>Standard Matrix</th></tr>
<tr><td>Reflection about the xy-plane
$T(x, y, z) = (x, y, -z)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$
$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$
$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$</td><td>$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$</td></tr>
<tr><td>Reflection about the xz-plane
$T(x, y, z) = (x, -y, z)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$
$T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$
$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$</td><td>$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Reflection about the yz-plane
$T(x, y, z) = (-x, y, z)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$
$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$
$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$</td><td>$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$</td></tr>
</table>

Table 3

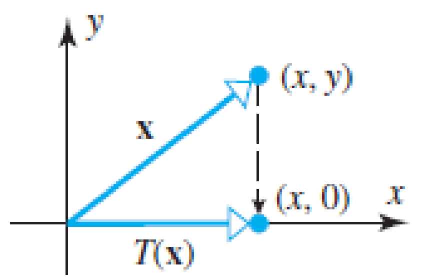
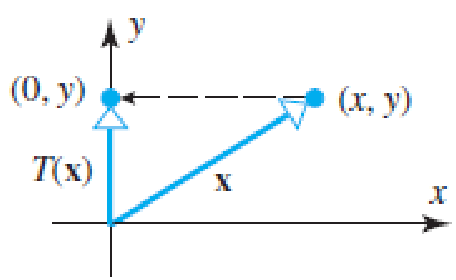
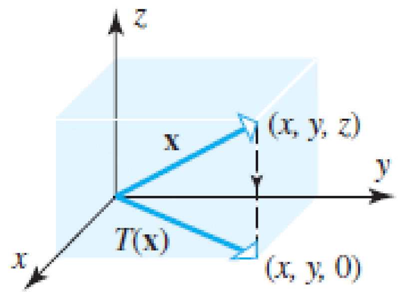
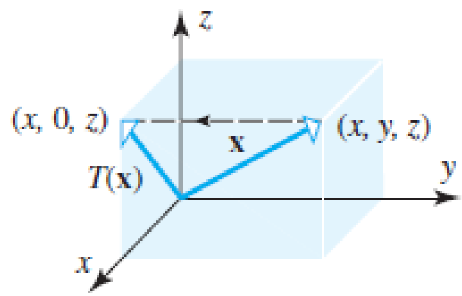
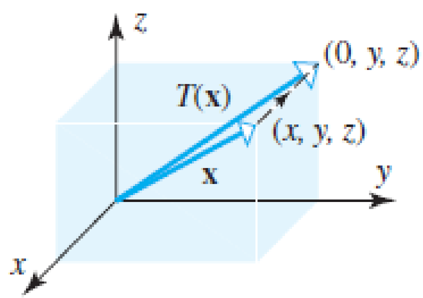
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection on the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Table 4

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection on the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators

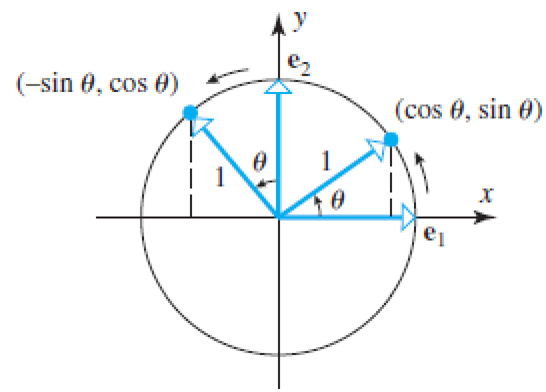


Table 5

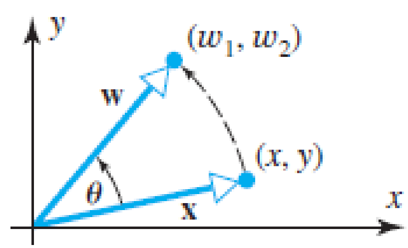
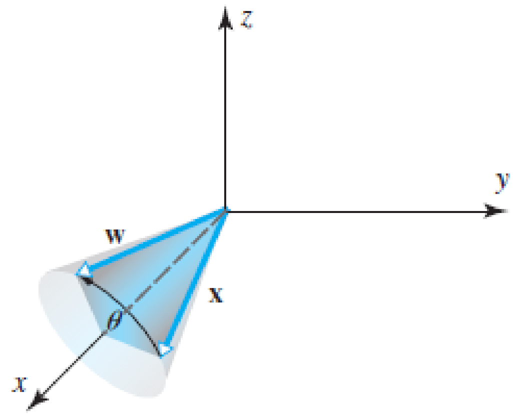
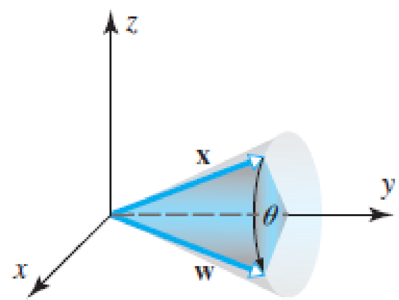
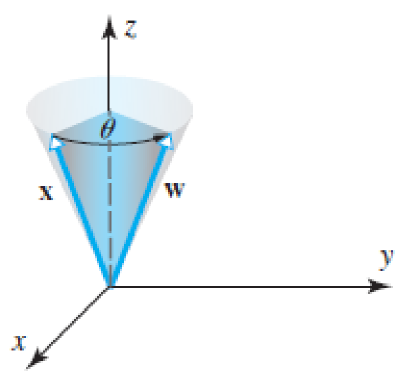
Operator	Illustration	Rotation Equations	Standard Matrix
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Table 6

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilations and Contractions

Table 7

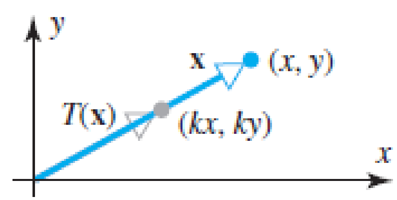
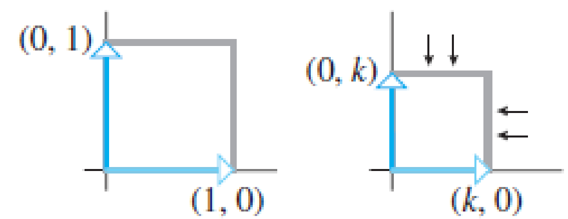
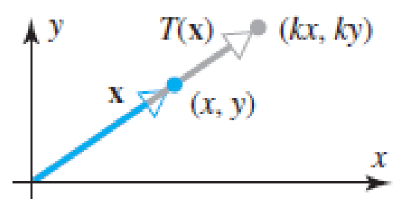
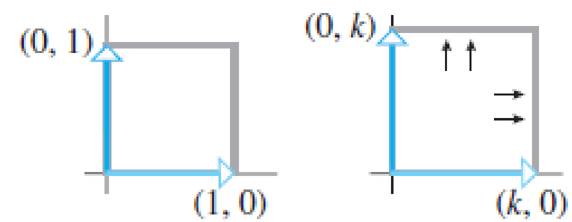
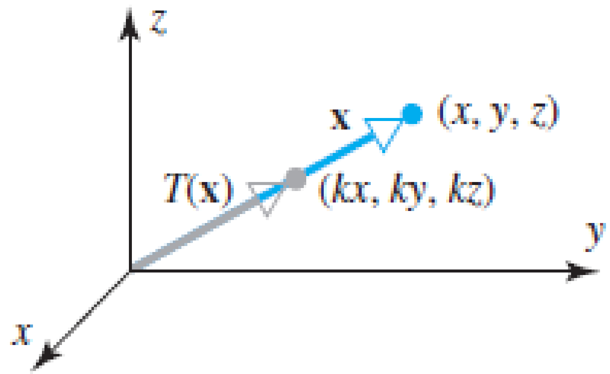
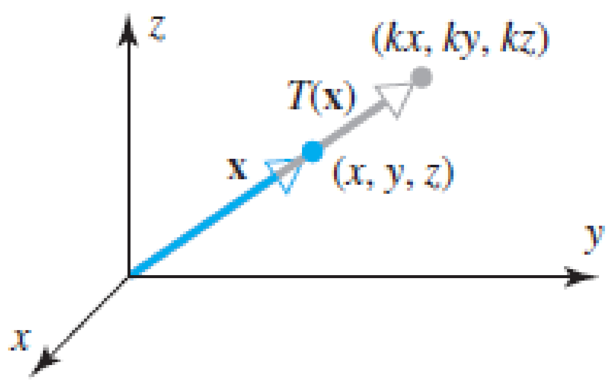
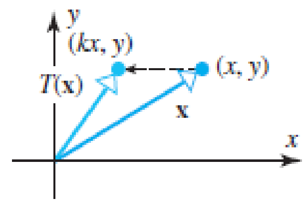
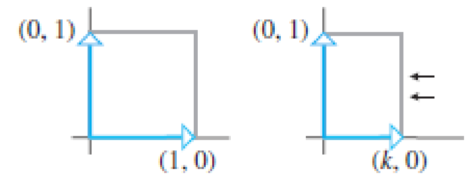
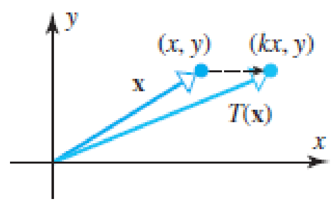
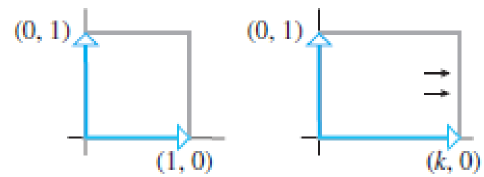
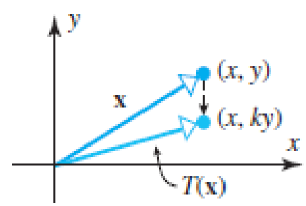
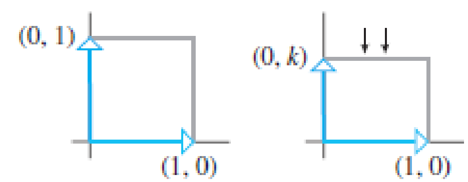
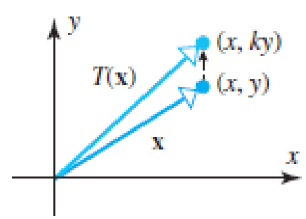
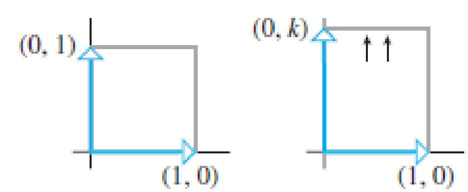
Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Standard Basis	Standard Matrix
Contraction with factor k on R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on R^2 $(k > 1)$			

Table 8

Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k on R^3 $(0 \leq k \leq 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on R^3 $(k \geq 1)$		

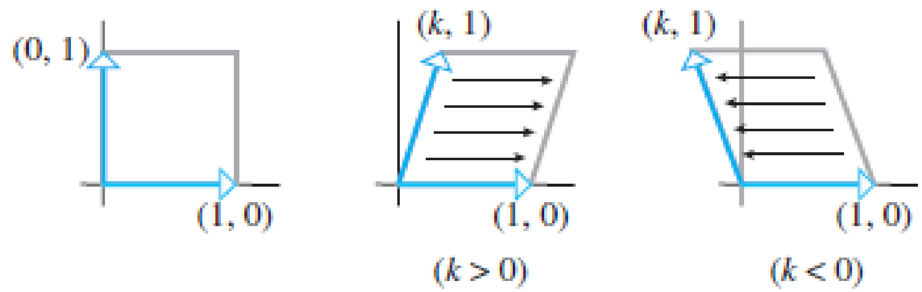
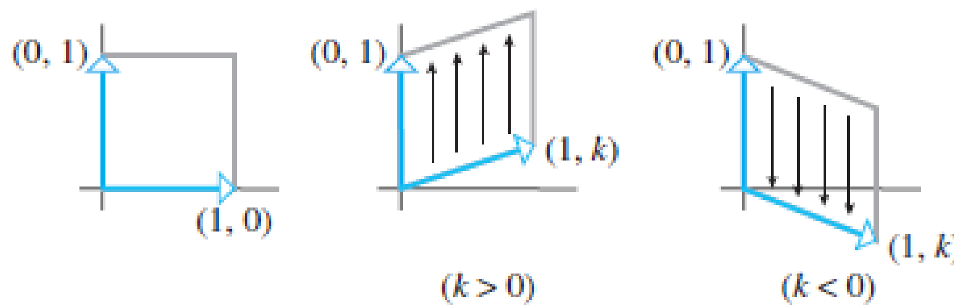
Expansion or Compression

Table 9

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Standard Basis	Standard Matrix
Compression of R^2 in the x-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of R^2 in the x-direction with factor k $(k > 1)$			
Operator	**Illustration $T(x, y) = (x, ky)$**	**Effect on the Standard Basis**	**Standard Matrix**
Compression of R^2 in the y-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of R^2 in the y-direction with factor k $(k > 1)$			

Shear

Table 10

Operator	Effect on the Standard Basis	Standard Matrix
Shear of R^2 in the x-direction with factor k $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of R^2 in the y-direction with factor k $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Section 4.10 Properties of Matrix Transformations

DEFINITION 1 A matrix transformation $T_A: R^n \to R^m$ is said to be ***one-to-one*** if T_A maps distinct vectors (points) in R^n into distinct vectors (points) in R^m.

THEOREM 4.10.1 *If A is an $n \times n$ matrix and $T_A: R^n \to R^n$ is the corresponding matrix operator, then the following statements are equivalent.*

(a) A is invertible.

(b) The range of T_A is R^n.

(c) T_A is one-to-one.

THEOREM 4.10.2 *$T: R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in R^n and for every scalar k:*

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

THEOREM 4.10.3 *Every linear transformation from R^n to R^m is a matrix transformation, and conversely, every matrix transformation from R^n to R^m is a linear transformation.*

THEOREM 4.10.4 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) A *is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of* A *is* I_n.

(*d*) A *is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of* A *are linearly independent.*

(*i*) *The row vectors of* A *are linearly independent.*

(*j*) *The column vectors of* A *span* R^n.

(*k*) *The row vectors of* A *span* R^n.

(*l*) *The column vectors of* A *form a basis for* R^n.

(*m*) *The row vectors of* A *form a basis for* R^n.

(*n*) A *has rank* n.

(*o*) A *has nullity* 0.

(*p*) *The orthogonal complement of the null space of* A *is* R^n.

(*q*) *The orthogonal complement of the row space of* A *is* $\{\mathbf{0}\}$.

(*r*) *The range of* T_A *is* R^n.

(*s*) T_A *is one-to-one.*

Section 4.11 Geometry of Matrix Operators on R^2

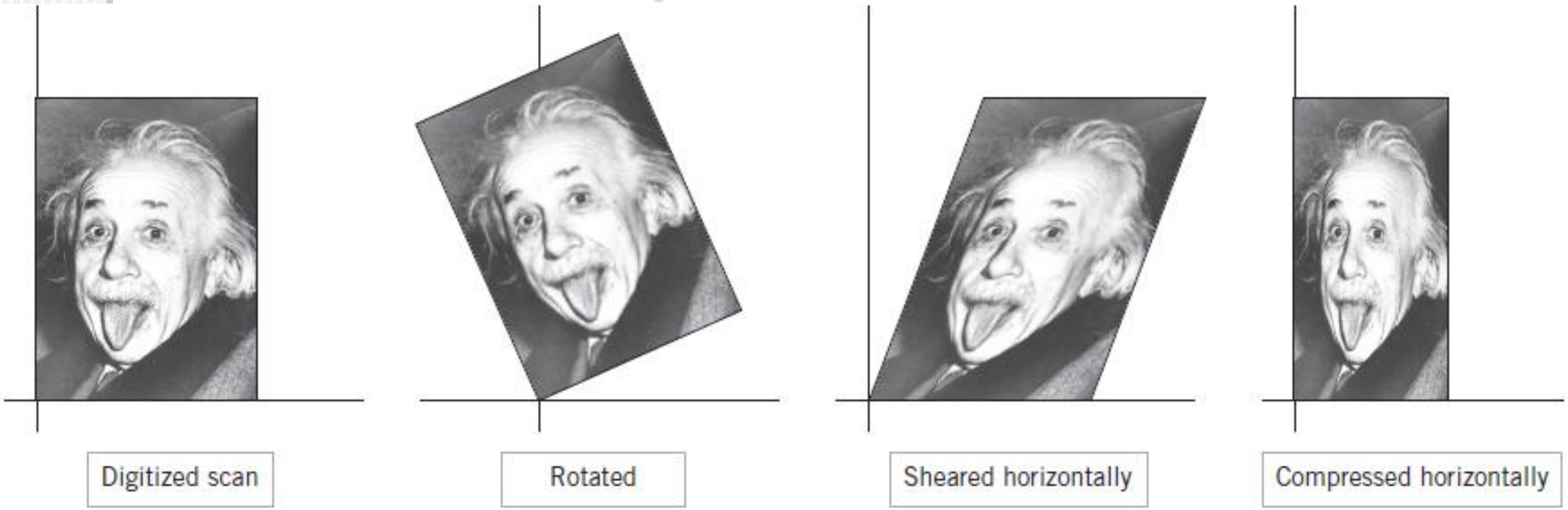
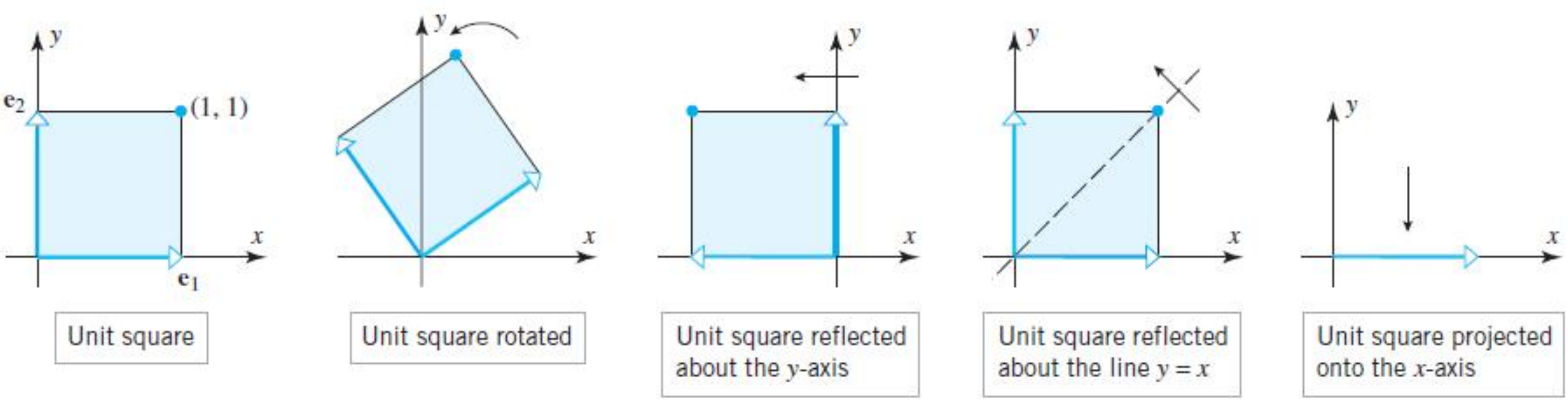


Figure 4.11.1



Matrix Operators

THEOREM 4.11.1 *If E is an elementary matrtix, then $T_E: R^2 \rightarrow R^2$ is one of the following:*

(a) *A shear along a coordinate axis.*

(b) *A reflection about $y = x$.*

(c) *A compression along a coordinate axis.*

(d) *An expansion along a coordinate axis.*

(e) *A reflection about a coordinate axis.*

(f) *A compression or expansion along a coordinate axis followed by a reflection about a coordinate axis.*

THEOREM 4.11.2 *If $T_A: R^2 \rightarrow R^2$ is multiplication by an invertible matrix A, then the geometric effect of T_A is the same as an appropriate succession of shears, compressions, expansions, and reflections.*

Computer Graphics

THEOREM 4.11.3 *If $T: R^2 \to R^2$ is multiplication by an invertible matrix, then:*

(a) *The image of a straight line is a straight line.*

(b) *The image of a straight line through the origin is a straight line through the origin.*

(c) *The images of parallel straight lines are parallel straight lines.*

(d) *The image of the line segment joining points P and Q is the line segment joining the images of P and Q.*

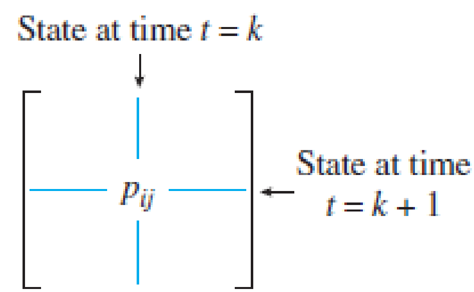
(e) *The images of three points lie on a line if and only if the points themselves lie on a line.*

Section 4.12 Dynamical Systems and Markov Chains

DEFINITION 1 A ***Markov chain*** is a dynamical system whose state vectors at a succession of time intervals are probability vectors and for which the state vectors at successive time intervals are related by an equation of the form

$$\mathbf{x}(k+1) = P\mathbf{x}(k)$$

in which $P = [p_{ij}]$ is a stochastic matrix and p_{ij} is the probability that the system will be in state i at time $t = k+1$ if it is in state j at time $t = k$. The matrix P is called the ***transition matrix*** for the system.



The entry p_{ij} is the probability that the system is in state i at time $t = k + 1$ if it is in state j at time $t = k$.

Regular Markov Chains

DEFINITION 2 A stochastic matrix P is said to be ***regular*** if P or some positive power of P has all positive entries, and a Markov chain whose transition matrix is regular is said to be a ***regular Markov chain***.

THEOREM 4.12.1 *If P is the transition matrix for a regular Markov chain, then:*

(*a*) *There is a unique probability vector* $\mathbf{q}$ *such that* $P\mathbf{q} = \mathbf{q}$.

(*b*) *For any initial probability vector* $\mathbf{x}_0$, *the sequence of state vectors*

$$\mathbf{x}_0, \quad P\mathbf{x}_0, \ldots, \quad P^k\mathbf{x}_0, \ldots$$

converges to $\mathbf{q}$.